Appendix C

Mathematical Details

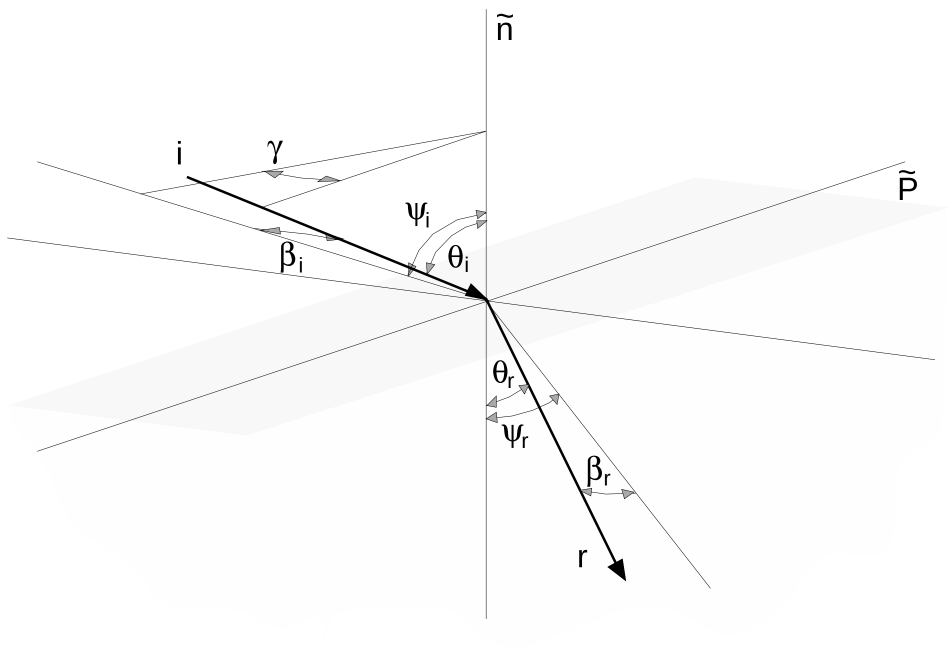

Figure C.1: Light ray at the prism surface.

C.1 Prism Matrix P

Consider what happens when the input ray, $\vec{R}_{in}$, varies by an arbitrary but small angle from the principal ray. Figure 5.1 shows how the chosen coordinate system, with the **Z** axis corresponding to the principal ray, relates to the surface of the prism.

Let

$$\begin{aligned} \psi_{i1} &= i_0 - \phi_{x.in} \\ \psi_{i2} &= i_0 + \phi_{x.out} \end{aligned} \tag{C.1}$$

Where i0 is the angle of incidence of the principal ray, and

$$\begin{aligned} \beta_{i1} &= \phi_{y.in} \\ \beta_{i2} &= \phi_{y.out} \end{aligned} \tag{C.2}$$

The surface normal, $\vec{n}$, and the vector $\vec{p}$ define the principal plane (principal section) of the prism. The input ray, i, refracted ray, r, and the normal, are also co-planar. This plane, however, is rotated about the normal by an angle of γ with respect to the principal plane. The various angles are related as follows:

$$\cos\theta = \cos\psi\cos\beta \tag{C.3}$$

$$\sin\beta = \sin\theta\sin\gamma \tag{C.4}$$

$$\sin\theta\cos\gamma = \sin\psi\cos\beta \tag{C.5}$$

$$\text{from Snell's law} \quad \sin\theta_i = n\sin\theta_r \tag{C.6}$$

$$\begin{aligned} \text{then} \quad \sin\beta_i &= \sin\theta_i\sin\gamma \\ &= n\sin\theta_r\sin\gamma \\ &= n\sin\beta_r \end{aligned} \tag{C.7}$$

Because β is the angle that the ray makes with the principle plane of the prism, $\beta_{r2} = \beta r1$ and from the definition of β in equation C.2 then:

$$\phi_{y.out} = \phi_{y.in} \tag{C.8}$$

From equation C.5

$$\begin{aligned}\sin\psi_{i1}\cos\beta_{i1} &= \sin\theta_{i1}\cos\gamma_1\\ &= n\sin\theta_{r1}\cos\gamma_1\\ &= n\sin\psi_{r1}\cos\beta_{r1}\end{aligned} \tag{C.9}$$

Applying to the first surface:

$$\sin\left(i_0-\phi_{x.in}\right)\cos\beta_i = n\sin\left(\frac{\alpha}{2}-\Delta\psi_r\right)\cos\beta_r \tag{C.10}$$

Expanding to give:

$$\left(\sin i_0-\phi_{x.in}\cos i_0\right)\cos\beta_i = n\left(\sin\frac{\alpha}{2}-\Delta\psi_r\cos\frac{\alpha}{2}\right)\cos\beta_r \tag{C.11}$$

Similarly at the second surface:

$$\left(\sin i_0+\phi_{x.out}\cos i_0\right)\cos\beta_i = n\left(\sin\frac{\alpha}{2}+\Delta\psi_r\cos\frac{\alpha}{2}\right)\cos\beta_r \tag{C.12}$$

But because β is small $\beta_i \approx n\beta_r$ and $\cos\beta \approx 1$. Adding equations C.11 and C.12

$$\begin{aligned}2\sin i_0+\left(\phi_{x.out}-\phi_{x.in}\right)\cos i_0 &\approx 2n\sin\frac{\alpha}{2}\\ \left(\phi_{x.out}-\phi_{x.in}\right)\cos i_0 &\approx 0\\ \phi_{x.out} &\approx \phi_{x.in}\end{aligned} \tag{C.13}$$

C.2 Prism Throughput

For light passing through a surface, from the Fresnel equations the **amplitude transmission coefficients** are:

$$t_{\perp} \equiv \left(\frac{E_{\perp t}}{E_{\perp i}}\right) = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)} \tag{C.14}$$

$$t_{\parallel} \equiv \left(\frac{E_{\parallel t}}{E_{\parallel i}}\right) = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} \tag{C.15}$$

For each of the polarisations the ratio of incident to transmitted intensity, or **transmittance**, for each surface is given by:

$$T = \left(\frac{n_t\cos\theta_t}{n_i\cos\theta_i}\right) t^2 \tag{C.16}$$

Considering both prism surfaces, where for the first surface $n_{i1} = 1$ (air) and $n_{t1} = n_{glass}$. At the second surface where the beam exits the prism $n_{i2} = n_{glass}$ and $n_{t2} = 1$. Additionally if the prism is used at minimum deviation (as is the case in this spectrograph) then $\theta_{i1} = \theta_{t2}\,(= \theta_{air})$ and $\theta_{t1} = \theta_{i2}\,(= \theta_{glass})$. The transmittance for the prism is therefore:

$$\begin{aligned} T_{prism} &= T_1 \times T_2 \\ &= t_1^2 \times t_2^2 \end{aligned} \tag{C.17}$$

This, however, will only give a value for each polarisation, so an average must be taken because the light entering the prism is unpolarised.

$$\begin{aligned} T_{prism} &= \frac{t_{\perp 1}^2 t_{\perp 2}^2 + t_{\parallel 1}^2 t_{\parallel 2}^2}{2} \\ &= \frac{\left(2\sin\theta_{glass}\cos\theta_{air}\right)^2 \left(2\sin\theta_{air}\cos\theta_{glass}\right)^2}{2\left(\sin\left(\theta_{air}+\theta_{glass}\right)\right)^4} \\ &\quad + \frac{\left(2\sin\theta_{glass}\cos\theta_{air}\right)^2 \left(2\sin\theta_{air}\cos\theta_{glass}\right)^2}{2\left(\sin\left(\theta_{air}+\theta_{glass}\right)\right)^4 \left(\cos\left(\theta_{air}-\theta_{glass}\right)\right)^4} \end{aligned} \tag{C.18}$$

For a 60° apex angle and $n_{glass} \approx 1.5$ then $\theta_{glass} = 30°$ and $\theta_{air} = 48.6°$

Substitution of these values into equation C.18 yields:

$$T_{prism} = 0.89 \tag{C.19}$$

C.3 Grating Matrix G

Following the definitions of Schroeder [1970] the diffraction equation for an echelle grating is:

$$m\lambda = d\left[\sin(\theta_b + \theta_i) + \sin(\theta_b - \theta_i)\right]\cos\gamma \tag{C.20}$$

The echelle is being used in a Quasi-Littrow configuration, so for our principal ray:

$$\theta_i = \theta_r = 0$$

giving

$$\frac{m\lambda}{d} = 2\sin\theta_b\cos\gamma \tag{C.21}$$

If we now consider an arbitrary input ray which differs from the principal ray by small angles $\Delta\theta_i$, $\Delta\gamma$. From equation C.20:

$$\sin(\theta_b - \Delta\theta_r) = \frac{m\lambda}{d\cos(\gamma + \Delta\gamma)} - \sin(\theta_b + \Delta\theta_i) \tag{C.22}$$

Expanding and substituting from equation C.21:

$$\begin{aligned}
-\Delta\theta_r\cos\theta_b &= \frac{m\lambda}{d(\cos\gamma - \Delta\gamma\sin\gamma)} - \frac{m\lambda}{d\cos\gamma} - \Delta\theta_i\cos\theta_b \\
&= \frac{m\lambda}{d}\left(\frac{\Delta\gamma\sin\gamma}{\cos\gamma(\cos\gamma - \Delta\gamma\sin\gamma)}\right) - \Delta\theta_i\cos\theta_b \\
&= \frac{m\lambda}{d}\left(\frac{\Delta\gamma\sin\gamma(\cos\gamma + \Delta\gamma\sin\gamma)}{\cos\gamma\left((\cos\gamma)^2 - (\Delta\gamma\sin\gamma)^2\right)}\right) - \Delta\theta_i\cos\theta_b \qquad \text{(C.23)}
\end{aligned}$$

Considering first order terns only, i.e. $(\Delta\gamma)^2 \approx 0$

$$\begin{aligned}
-\Delta\theta_r\cos\theta_b &\approx \frac{m\lambda}{d}\left(\frac{\Delta\gamma\sin\gamma\cos\gamma}{(\cos\gamma)^3}\right) - \Delta\theta_i\cos\theta_b \\
\Delta\theta_r &= \Delta\theta_i - \frac{m\lambda}{d}\frac{\Delta\gamma\tan\gamma}{\cos\gamma\cos\theta_b} \qquad \text{(C.24)}
\end{aligned}$$

But substituting from equation C.21:

$$\begin{aligned}
-\Delta\theta_r &= \Delta\theta_i - 2\sin\theta_b\cos\gamma\frac{\Delta\gamma\tan\gamma}{\cos\gamma\cos\theta_b} \\
&= \Delta\theta_i - \Delta\gamma\tan\gamma\tan\theta_b \qquad \text{(C.25)}
\end{aligned}$$

C.4 Grating Through-put

The intensity distribution, I, across the blaze peak is determined by the effective width of a single echelle groove. This is given by:

$$I(\delta) = \left(\frac{\sin\delta}{\delta}\right)^2 \tag{C.26}$$

where δ is the phase difference between the centre and edge of a single groove of effective width D. For an echelle with a groove spacing of d, in quasi Littrow mode

$$D = d\cos\theta_b \tag{C.27}$$

and

$$\delta = \frac{2\pi}{\lambda}D\frac{\sin\theta_r}{2} \tag{C.28}$$

$$= \frac{\pi d}{\lambda}\sin\theta_r\cos\theta_b \tag{C.29}$$

To find the efficiency of an echelle grating it is necessary to calculate the distribution of light of a given wavelength across all orders for which $|\theta_r| < 90°$. The efficiency can then be calculated for each wavelength by

$$\varepsilon_\lambda = \frac{I_{m0}}{\sum\limits_m I_m} \tag{C.30}$$

where I_{m0} = the relative intensity of light in the order being used and I_m = the relative intensity of light in order m.

Referring back to equation C.20, for positive values of θ_r there are no problems, however for $\theta_r < 0$ part of the light will undergo a second reflection as shown in Figure C.2.

The fraction of the light that is reflected a second time, R, can be shown to be:

$$\begin{aligned} R &= \tan(-\theta_r)\frac{d\sin\theta_b}{\cos\gamma}\frac{1}{d\cos\theta_b} \\ &= \frac{\tan(-\theta_r)\tan\theta_b}{\cos\gamma} \end{aligned} \tag{C.31}$$

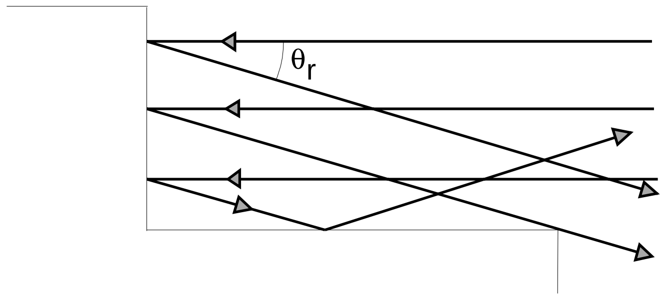

Figure C.2: Diffracted light with partial secondary reflection.

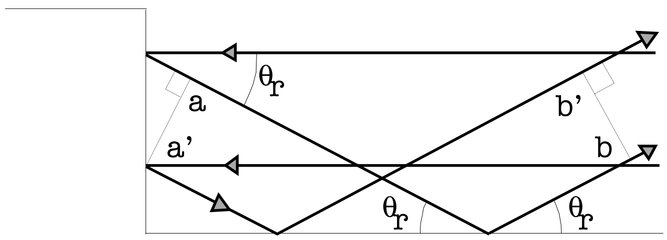

Figure C.3: Double reflection in a single groove.

The value of R will reach a maximum of 1 when:

$$\theta_r = -\arctan\left(\frac{\cos\gamma}{\tan\theta_b}\right) \tag{C.32}$$

For values of θ_r equal to or more negative than this value **all** of the light undergoes a second reflection.

Figure C.3 shows a single groove where the light undergoes double reflection.

Clearly the paths $\underline{ab}$ and $\underline{a'b'}$ are the same length, so the additional reflection does not cause any change in the relative phase difference between rays. Equation C.26, therefore, still remains valid.

For the light entering two adjacent grooves, as shown in Figure C.4, the length of $\underline{cd} = \underline{c'd'}$. It is therefore possible to ignore the fact that a second reflection has taken place and treat the diffracted rays as if they had a positive value of θ_r.

Once any secondary reflections are taken into account equation C.30 can be

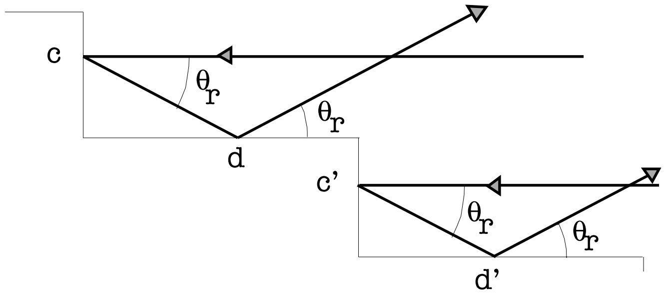

Figure C.4: Double reflection in adjacent grooves.

used to calculate the theoretical efficiency over the whole spectrum.

C.5 Exposure times

Consider now photons arriving at the detector with a wavelength range of λ to $\lambda + \Delta\lambda$. The area of the detector effected by these photons depends on the “width" of the orders and the dispersion of the grating. The width of the orders can be found by calculating the profile of the fibre image in the cross-dispersion direction in a similar way to that previously done for the dispersion direction (see Figure 7.1 in Chapter 7). From the discussion of the fibre image in Chapter 6, the width, W, of the elliptical shaped profile will be:

$$W = \frac{F_{0.cam}}{F_{0.coll}} \times 2r_f \tag{C.33}$$

Because it is the peak value that is of interest then a more appropriate “effective" width may be estimated by taking the width of a rectangular profile which has the same area under the curve and the same peak value.

$$= \frac{\pi}{4} W = \frac{\pi F_k}{2F_c} \times r_f \tag{C.34}$$

Rearranging equation 7.4 (Chapter 7) gives the length of the illuminated area of the detector as:

$$l_{det} = \frac{2F_k \tan\theta_b}{\lambda} \times \Delta\lambda \tag{C.35}$$

The area of the detector effected is therefore:

$$a_{det} = l_{det} \times w_{det} = \frac{\pi F_k^2 \tan\theta_b}{\lambda F_c} \times \Delta\lambda \times r_f \tag{C.36}$$

Taking into account the spectrograph efficiency, ε, the electron density in the detector, N, for n photons entering the optical fibre link is:

$$N_{det}(\lambda) = \frac{n(\lambda)}{a_{det}(\lambda)} \times \varepsilon_{spect}(\lambda) \times C_{FRD} \tag{C.37}$$

The spectral distribution of the light from a star can be approximated using “black body" radiation curves. The energy emitted per unit surface area by a black-body at temperature T, in the wavelength range of λ to $\lambda + \Delta\lambda$ is given by the Planck equation:

$$E(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{\Delta\lambda}{e^{\frac{hc}{\lambda kT}} - 1} \ \mathrm{J/sec/m^2} \tag{C.38}$$

The Stefan - Boltzman law gives the total energy output per unit surface area of the black-body (star in our case):

$$\begin{aligned} E_{total}(T) &= \int_0^\infty E(\lambda, T) \\ &= \sigma T^4 \\ &= \frac{2\pi^5}{15} hc^2 \left(\frac{hc}{kt}\right)^{-4} \ \mathrm{J/sec/m^2} \end{aligned} \tag{C.39}$$

Using this, the fraction of the energy emitted at a given wavelength (range) is then:

$$E_{rel}(\lambda, T) = \frac{15}{\pi^4 \lambda^5} \left(\frac{hc}{kt}\right)^4 \frac{\Delta\lambda}{e^{\frac{hc}{\lambda kT}} - 1} \tag{C.40}$$

Energy from the sun arriving at the top of the earths atmosphere, L_{sun}, is 1350 $\mathrm{J/sec/m^2}$. This corresponds to an apparent bolometric magnitude, m_{bol}, of -26.9. Therefore at $m_{bol} = 0$:

$$L_0 = L_{sun} 10^{\left(\frac{M_{bol,sun}}{2.5}\right)} = 24 \times 10^{-9} \ \mathrm{J/sec/m^2} \tag{C.41}$$

On the assumption that atmospheric absorption is negligible in the wavelength range of interest, then the number of photons entering a telescope of aperture A can be calculated for a star with an apparent bolometric magnitude, m_{bol}, and an effective surface temperature, T.

$$n_{star}(\lambda, T) = 10^{\left(\frac{M_{bol}}{2.5}\right)} L_0 E_{rel}(\lambda, T) \frac{\lambda}{hc} \frac{\pi A^2}{4} \text{ photons / sec} \tag{C.42}$$

then the number of electrons generated in a single pixel is:

$$N_{pixel}(\lambda) = 10^{\left(\frac{M_{bol}}{2.5}\right)} L_0 E_{rel}(\lambda, T) \frac{a_{pixel}}{a_{det}(\lambda)} \frac{\lambda}{hc} \frac{\pi A^2}{4} \varepsilon_{spect}(\lambda) C_{FRD} \text{ e}^- \text{ / sec} \tag{C.43}$$